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DUFFIN-KEMMER-PETIAU EQUATION WITH NONMINIMAL COUPLING TO CURVATURE

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The generalized Duffin-Kemmer-Petiau equation in curved space-time is proposed for non-minimal coupling to the curvature and external fields. The corresponding scalar and vector fields equation are found. Equations are presented, which are equivalent to those of a scalar field with conformal coupling and electromagnetic field with non-minimal coupling to the curvature. The gauge-invariant Duffin-Kemmer-Petiau equation with non-minimal coupling is given.

1. Introduction

The Duffin-Kemmer-Petiau (DKP) equation is a first-order relativistic wave equation for spin 0 and 1 bosons [1]. Recently there has been an increasing interest to DKP theory in external fields and curved space-time (see [2] and references there). For minimal coupling to the curvature or an external vector field, the DKP equation in the scalar case is equivalent to the corresponding Klein-Gordon-Fock equation [2, 3]. For the vector case, the DKP equation with minimal coupling is equivalent to the Maxwell or Proca equations.

In the DKP formalism, a wave function is multicomponent. That is why the simplest non-minimal interactions with external fields have a more complicated structure than in usual formalism. It has applications in describing of interactions of mesons with nuclei [4], for studies of pionic atoms [5] etc. The question on non-minimal coupling to the curvature in the DKP formalism was not considered earlier, with the exception of the work [6]. However, in this work, the DKP equation with conformal coupling was written only using an auxiliary field obeying contradictory conditions: this field is vector for general coordinate transformation and is constructed only from the metric tensor and its derivatives.

In the present work, a generalized DKP equation is introduced, with non-minimal coupling to the curvature and external fields, and the corresponding scalar and vector field equations are found. We use the system of units where $\hbar = c = 1$. The signs of the curvature tensor and the Ricci tensor are chosen such that $R^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^i_{nl} \Gamma^n_{jk} - \Gamma^i_{nk} \Gamma^n_{jl}$ and $R_{ik} = R^l_{ilk}$, $R = R^k_k$, where Γ^i_{jk} are Christoffel symbols.

2. DKP formalism

The DKP equation for a field with mass m is given by

$$(i\beta^a \partial_a - m)\psi = 0, \quad (1)$$

where the matrices β^a obey the algebraic relations

$$\beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = \beta^a \eta^{bc} + \beta^c \eta^{ba}, \quad (2)$$

and η^{ab} is a constant diagonal matrix (in particular, $\text{diag}(1, -1, \dots, -1)$). For the space-time dimension $N = 4$, the DKP algebra (2) has 5- (corresponding to spin 0), 10- (corresponding to spin 1) and 1-dimensional (trivial) irreducible representations.

Each component of the free massive DKP field obeys the Klein-Gordon-Fock equation, because

$$d(\partial)(i\beta^a \partial_a - m) = -(\partial_a \partial^a + m^2)I, \quad (3)$$

where I is the identity matrix and

$$d(\partial) = mI + i\beta^a \partial_a + (2\eta^{ab} - \beta^a \beta^b - \beta^b \beta^a) \frac{\partial_a \partial_b}{2m}. \quad (4)$$

The DKP equation for the massless case was written by Harish-Chandra [7]

$$i\beta^a \partial_a \psi - \gamma \psi = 0. \quad (5)$$

Here γ is a singular matrix satisfying

$$\beta^a \gamma + \gamma \beta^a = \beta^a, \quad \gamma^2 = \gamma. \quad (6)$$

If γ is a solution to (6), then $1 - \gamma$ is also one. The following theorem takes place [7]: for any irreducible set of matrices β^a there exist either none or just two such matrices γ .

We introduce Umezawa projectors [8], generalized to N -dimensional case, and an arbitrary diagonal matrix (η^{ab}):

$$P = \det(\eta_{ab})(\beta^0)^2(\beta^1)^2 \dots (\beta^{N-1})^2, \quad P^a = P\beta^a, \quad (7)$$

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$$Q^a = \det(\eta_{ab}) (\beta^1)^2 \dots (\beta^{N-1})^2 [\eta^{a0} - \beta^a \beta^0], \quad (8)$$

$$Q^{ab} = Q^a \beta^b. \quad (9)$$

Under the infinitesimal Lorentz transformations

$$x'^a = \Lambda^a_b x^b, \quad \Lambda^a_b = \delta^a_b + \omega^a_b, \quad \omega_{ab} = -\omega_{ba} \quad (10)$$

we have $\psi'(x') = S(\Lambda)\psi(x)$,

$$S(\Lambda) = 1 + \frac{1}{2} \omega_{ab} S^{ab}, \quad S^{ab} = \beta^a \beta^b - \beta^b \beta^a, \quad (11)$$

and $P\psi$ transforms as scalar, $P^a\psi$ and $Q^a\psi$ as vectors, and $Q^{ab}\psi$ as antisymmetric tensor of second rank.

The DKP equation for curved space is generalized [2] analogously to the Fock-Ivanenko-Weyl method [9] for the Dirac equation. On a space-time manifold, a sets of N vector fields $e^i_{(a)}(x)$ and $e^{(b)i}(x)$ (tetrads in the four-dimensional case) are introduced with the relations

$$e^i_{(a)} e_{(b)i} = \eta_{ab}, \quad e^i_{(a)} e_i^{(b)} = \delta_a^b. \quad (12)$$

The covariant derivative of the DKP field ψ is defined by

$$\nabla_k \psi = (\partial_k + \frac{1}{2} \omega_{kab} S^{ab}) \psi, \quad (13)$$

where the “spin” connection ω_{kab} obeys the relations

$$\omega_{kab} = e_{(a)l} e^l_{(b)} \Gamma^l_{jk} - e^j_{(b)} \partial_k e_{(a)j}, \quad \omega_{kab} = -\omega_{kba}. \quad (14)$$

The DKP equations in curved space-time, for the massive and massless fields, are written as

$$i\beta^k \nabla_k \psi - m\psi = 0, \quad i\beta^k \nabla_k \psi - \gamma\psi = 0, \quad (15)$$

where $\beta^k = e^k_{(a)} \beta^a$. There equations are equivalent to those of minimally coupled scalar or vector fields [2, 10].

3. Nonminimal coupling to curvature

Usually, in quantum theory in curved space-time, a scalar field is considered with the Lagrangian

$$L(x) = \sqrt{|g|} [\partial_i \varphi^* \partial^i \varphi - (m^2 + V_g) \varphi^* \varphi - U(\varphi^* \varphi)] \quad (16)$$

and the corresponding equation of motion

$$(\nabla^i \nabla_i + V_g + m^2 + U'(\varphi^* \varphi)) \varphi(x) = 0, \quad (17)$$

where ∇_i are covariant derivatives in the metric g_{ik} , $g = \det(g_{ik})$, $U(\varphi^* \varphi)$ is self-interaction, $V_g = 0$ for minimal coupling to curvature. In the case $V_g = \xi_c R$, where $\xi_c = (N-2)/[4(N-1)]$ (conformal coupling), the massless equation is conformally invariant if $U = \lambda(\varphi^* \varphi)^{N/(N-2)}$, $\lambda = \text{const}$. For V_g of arbitrary form, the third and fourth derivatives can appear in the metric energy-momentum tensor and the Einstein equations. However, for the Gauss-Bonnet-type coupling [11]

$$V_g = \xi R + \chi(R_{lmnp} R^{lmnp} - 4R_{lm} R^{lm} + R^2) \quad (18)$$

the energy-momentum tensor does not contain higher than the second-order derivatives of the metric.

The electromagnetic field with the tensor $F^{ik} = \partial^i A^k - \partial^k A^i$ and a minimal coupling to curvature has the equation

$$\nabla_i F^{ik} = 0, \quad (19)$$

which is conformal invariant if $N = 4$. For a vector field with nonminimal coupling to the curvature, the equation is often chosen [12]-[17] in the form

$$\nabla_i \left((1 - \lambda_1 R) F^{ik} - \lambda_2 (R^i_n F^{nk} - R^n_k F^{ni}) - \lambda_3 R^{iklm} F_{lm} \right) + (\xi_1 R + m^2) A^k + \xi_2 R^k_i A^i = 0, \quad (20)$$

where $\lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2$ are constants. The corresponding Lagrangian is

$$L_v = -\frac{1}{4} \sqrt{|g|} F_{ik} F^{ik} + L_I + L_{II}, \quad (21)$$

where

$$L_I = \frac{\sqrt{|g|}}{4} [\lambda_1 R F_{ik} F^{ik} + 2\lambda_2 R_{ik} F^{in} F^k_n + \lambda_3 R_{iklm} F^{ik} F^{lm}], \quad (22)$$

$$L_{II} = \frac{\sqrt{|g|}}{2} [(\xi_1 R + m^2) A_i A^i + \xi_2 R_{ik} A^i A^k]. \quad (23)$$

The additional terms from (23) broke the gauge invariance of the theory. The additional terms from (22) can produce a lot of new effects: photons creation in expanding Universe [14] (for Eq. (19), photons do not create due to conformal invariance); polarization dependence of the photon trajectory [16]; variation of the speed of light in curved space [17], etc.

4. DKP equation with non-minimal coupling

We consider the generalized DKP equations for non-minimal coupling to the curvature and external fields:

$$i\beta^k (\nabla_k + iB_k) \left(1 + \gamma \sum \zeta_{k_1 \dots k_p} \beta^{k_1} \dots \beta^{k_p} \right) \psi - \alpha \gamma \psi - \sum V_{k_1 \dots k_q} \beta^{k_1} \dots \beta^{k_q} (1 - \gamma) \psi = 0, \quad (24)$$

where $\alpha = \text{const} \neq 0$, $B_k(x)$ is an external vector field, $\zeta_{k_1 \dots k_p}(x)$, $V_{k_1 \dots k_q}(x)$ ($p, q = 0, 1, 2, \dots$) are arbitrary external fields, e.g., R , R_{ik} , R_{iklm} . To find the scalar equation corresponding to Eq. (24), we use the DKP algebra and the following relations: $P^i \beta^j = g^{ij} P$,

$$P \nabla_i \psi = \nabla_i (P \psi) = \partial_i (P \psi) = P \partial_i \psi, \quad (25)$$

$$P^i \nabla_k \psi = \nabla_k (P^i \psi) = \partial_k (P^i \psi) + \Gamma^i_{lk} P^l \psi, \quad (26)$$

$$P \sum V_{k_1 \dots k_q} \beta^{k_1} \dots \beta^{k_q} = \sum_{\text{even}} V_{k_2 \dots k_q}^{k_2} P + \sum_{\text{odd}} V_{k_1 \dots k_{q-2} k_q}^{k_1} P^{k_q}, \quad (27)$$

$$P^k \sum \zeta_{k_1 \dots k_p} \beta^{k_1} \dots \beta^{k_p} = \sum_{\text{even}} \zeta_{k_2 \dots k_{p-2} k_p}^{k_2} P^{k_p} + \sum_{\text{odd}} \zeta_{k_3 \dots k_p}^{k_3} P^{k_p}, \quad (28)$$

where $P^i = e_{(a)}^i P^a$, \sum_{even} and \sum_{odd} denote summation with even and odd number of indices. Multiplying (24) by $(1-\gamma)P$ and $(1-\gamma)P^l$ and taking into account (25)–(28), and the equalities following from (6)

$$\gamma(1-\gamma) = 0, \quad \gamma\beta^k = \beta^k(1-\gamma), \quad \gamma\beta^i\beta^k = \beta^i\beta^k\gamma, \quad (29)$$

$$\gamma P = P\gamma, \quad \gamma P^k = P^k(1-\gamma), \quad (30)$$

we obtain

$$(1-\gamma) \left[iD_k \left(\left(\delta_l^k + \sum_{\text{even}} \zeta_{k_2 \dots k_q}^{k_2} \right) P^l + \sum_{\text{odd}} \zeta_{k_3 \dots k_p}^{k_3} P \right) - \sum_{\text{even}} V_{k_2 \dots k_q}^{k_2} P \right] \psi = 0, \quad (31)$$

$$(1-\gamma)P^l\psi = \frac{1-\gamma}{\alpha} \left[iD^l - \sum_{\text{odd}} V_{k_3 \dots k_q}^{k_3} \right] P\psi, \quad (32)$$

where $D_k = \nabla_k + iB_k$. Substituting (32) in (31), one obtains an equation for the scalar component $\phi \equiv (1-\gamma)P\psi$:

$$\left\{ D_k \left[\left(g^{kl} + \sum_{\text{even}} \zeta_{k_3 \dots k_{p-1} l}^{k_3} \right) \left(D_l + i \sum_{\text{odd}} V_{l k_3 \dots k_q}^{k_3} \right) \right] - i\alpha D_k \left(\sum_{\text{odd}} \zeta_{k_3 \dots k_p}^{k_3} \right) + \alpha \sum_{\text{even}} V_{k_2 \dots k_q}^{k_2} \right\} \phi = 0. \quad (33)$$

For a scalar representation of the DKP algebra, this equation is equivalent to (24). The DKP equation corresponding to Eq. (17) may be written as

$$i\beta^k \nabla_k \psi - \alpha \gamma \psi - \frac{1}{\alpha} (m^2 + V_g P + U' P) (1-\gamma) \psi = 0, \quad (34)$$

where $U' = U'(\alpha^{-1}\psi^+ P^+ (1-\gamma^+) (1-\gamma) P \psi)$. If $m \neq 0$, then the choice $\alpha = m$ gives identical dimensions of different components of the DKP field. For the massless case we take $\alpha = 1$, then the dimensions of $P\psi$ and $P^k\psi$ are different.

For spin 1 we take into account that

$$Q^{ik} = -Q^{ki}, \quad Q^i \beta^k \beta^l = Q^{ik} \beta^l = Q^i g^{kl} - Q^k g^{il} \quad (35)$$

and introduce the tensors E_{mn}^{lk} , O_n^{lk} , H_n^l , G_n^{lk} :

$$Q^{lk} \left(1 + \sum_{\text{even}} \zeta_{k_1 \dots k_p} \beta^{k_1} \dots \beta^{k_p} \right) = E_{mn}^{lk} Q^{mn}, \quad (36)$$

$$Q^{lk} \sum_{\text{odd}} \zeta_{k_1 \dots k_p} \beta^{k_1} \dots \beta^{k_p} = O_n^{lk} Q^n, \quad (37)$$

$$Q^l \sum_{\text{even}} V_{k_1 \dots k_q} \beta^{k_1} \dots \beta^{k_q} = H_n^l Q^n, \quad (38)$$

$$Q^{lk} \sum_{\text{odd}} V_{k_1 \dots k_q} \beta^{k_1} \dots \beta^{k_q} = G_n^{lk} Q^n. \quad (39)$$

Multiplying (24) by $(1-\gamma)Q^l$ and $(1-\gamma)Q^{ln}$, one obtains

$$(1-\gamma) \left[iD_k (E_{mn}^{lk} Q^{mn} + O_n^{lk} Q^n) - H_n^l Q^n \right] \psi = 0, \quad (40)$$

$$(1-\gamma)Q^{ln}\psi = \frac{1-\gamma}{\alpha} \left[i(D^n Q^l - D^l Q^n) - G_m^{ln} Q^m \right] \psi. \quad (41)$$

We denote $A^k = (1-\gamma)Q^k\psi$,

$$F^{ik} = (D^i Q^k - D^k Q^i) (1-\gamma)\psi = D^i A^k - D^k A^i,$$

and substitute (41) in (40). As a result, we have

$$D_k (E_{mn}^{lk} F^{nm}) + [iD_k (E_{mn}^{lk} G_p^{mn} - \alpha O_p^{lk}) + \alpha H_p^l] A^p = 0. \quad (42)$$

For a vector representation of the DKP algebra, this equation is equivalent to (24). In particular, for a vector representation, the following DKP equation:

$$i\beta^k \nabla_k \left(1 - \gamma \left(\lambda_1 R + \lambda_2 R_{mn} \beta^m \beta^n + \frac{\lambda_3}{2} (R_{mn} \beta^m \beta^n + R_{mnpq} \beta^m \beta^p \beta^n \beta^q) \right) \right) \psi - \alpha \gamma \psi - \frac{1}{\alpha} \left(m^2 + \xi_1 R + \xi_2 (R - R_{mn} \beta^m \beta^n) \right) (1-\gamma) \psi = 0 \quad (43)$$

reproduces Eq. (20) with non-minimal coupling.

The DKP equation

$$i\beta^k \nabla_k \left(1 + \gamma \sum \zeta_{k_1 \dots k_{2n}} \beta^{k_1} \dots \beta^{k_{2n}} \right) \psi - \gamma \psi = 0 \quad (44)$$

is invariant under the transformation

$$\psi \rightarrow \psi + (1-\gamma)\Phi, \quad \text{if } \gamma\beta^k \nabla_k \Phi = 0. \quad (45)$$

In a vector representation, such transformations correspond to addition of derivatives of some scalar function to the components $A_k = (1-\gamma)Q_k\psi$ [10], i.e., Eq. (44) corresponds to the non-minimally coupled gauge-invariant vector field.

The DKP equation of the form

$$i\beta^k \nabla_k \psi - \gamma \psi - \xi_c R P (1-\gamma) \psi = 0 \quad (46)$$

describes the conformally invariant scalar field in a scalar representation of the DKP algebra and the minimally coupled electromagnetic field (19), for a vector representation ($P = 0$).

For DKP algebra representations in which

$$(\beta^0)^+ = \beta^0, \quad (\beta^\nu)^+ = -\beta^\nu, \quad \gamma^+ = \gamma, \quad (47)$$

where $\nu = 1, \dots, N-1$, the Lagrangian for the DKP equation can be written similarly to that for the Dirac equation. For examples, the Lagrangian corresponding to Eq. (34) is

$$L = \sqrt{|g|} \left[i\bar{\psi}\gamma\beta^k\nabla_k\psi - i\nabla_k\bar{\psi}\beta^k\gamma\psi - \alpha\bar{\psi}\gamma\psi - \right. \\ \left. - \alpha^{-1}\bar{\psi}(m^2 + V_g P)(1-\gamma)\psi - U(\alpha^{-1}\bar{\psi}(1-\gamma)P\psi) \right], \quad (48)$$

where $\bar{\psi} = \psi^+\eta^0$ is the DKP conjugate function, $\eta^0 = 2(\beta^0)^2/\eta^{00} - 1$,

$$\nabla_k\bar{\psi} = \partial_k\bar{\psi} - \frac{1}{2}\omega_{kab}\bar{\psi}S^{ab}. \quad (49)$$

Thus the DKP formalism is an equivalent form for the description of scalar and vector fields with various types of coupling to the curvature and external fields (see Eqs. (24), (33), (42) and the particular cases (34), (43), (44), (46)). Taking into account a possible non-minimal coupling of the scalar and vector fields to curvature may be important for the early Universe. The questions concerning the type of coupling to the curvature pertain to experiment. The generalized DKP equations can be used in those problems, where the general covariance is required and the formalism of first-order differential equations is preferable.

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References

- [1] G. Petiau, University of Paris thesis (1936), published in *Acad. Roy. de Belg., A. Sci. Mem. Collect* **16**, N 2, 1 (1936);
R.J. Duffin, *Phys. Rev.* **54**, 1114 (1938);
N. Kemmer, *Proc. Roy. Soc. A* **173**, 91 (1939).
- [2] J.T. Lunardi, B.M. Pimentel, and R.G. Teixeira, in: "Geometrical Aspects of Quantum Fields", eds. A.A. Bytsenko, A.E. Goncalves, and B.M. Pimentel, World Scientific, Singapore, 2001, p. 111, gr-qc/9909033.
- [3] V.Ya. Fainberg and B.M. Pimentel, *Theor. Math. Phys.*, **124**, 1234 (2000), hep-th/9911219; *Phys. Lett.*, **A 271**, 16 (2000).
- [4] B.C. Clark et al., *Phys. Rev. Lett.* **55**, 592 (1985);
L.K. Kerr et al., *Prog. Theor. Phys.* **103**, 321 (2000).
- [5] G. Kalbermann, *Phys. Rev. C* **34**, 2240 (1986).
- [6] R. Casana, B.M. Pimentel, J.T. Lunardi, and R.G. Teixeira, *Class. Quantum Grav.* **22**, 3083 (2005), gr-qc/0311042.
- [7] Harish-Chandra, *Proc. Roy. Soc.*, **A 186**, 502 (1946).
- [8] H. Umezawa, "Quantum Field Theory", North-Holland Publ., Amsterdam, 1956.
- [9] V. Fock and D. Iwanenko, *Compt. Rend. Acad. Sci. Paris* **188**, 1470 (1929);
H. Weyl, *Z. Phys.* **56**, 330 (1929).
- [10] R. Casana, B.M. Pimentel, J.T. Lunardi, and R.G. Teixeira, *Int. J. Mod. Phys. A* **17**, 4197 (2002).
- [11] Yu.V. Pavlov, *Theor. Math. Phys.* **140**, 1095 (2004), gr-qc/0409009; *Grav. & Cosmol.* **11**, No. 1-2, 123 (2005).
- [12] M. Novello and J.M. Salim, *Phys. Rev. D* **20**, 377 (1979).
- [13] P.C.W. Davies and D.J. Toms, *Phys. Rev. D* **31**, 1363 (1985).
- [14] M. Novello, L.A.R. Oliveira, and J.M. Salim, *Class. Quantum Grav.* **7**, 51 (1990).
- [15] R. Lafrance and R.C. Myers, *Phys. Rev. D* **51**, 2584 (1995).
- [16] A.R. Prasanna and S. Mohanty, *Class. Quantum Grav.* **20**, 3023 (2003).
- [17] P. Teyssandier, *Annales Fond. Broglie* **29**, 173 (2004), gr-qc/0303081.